Quantum Transformation Groupoids in the Setting of Operator Algebras

Thomas Timmermann

24.01.2013

Westfälische Wilhelms-Universität Münster

When are crossed products quantum transformation groupoids?

(exposé abrégé)

dynamical system $\Gamma \circlearrowright X$ ⟶ **groupoid** $X \rtimes \Gamma$

groupoid algebra / function algebra

$\Gamma \circlearrowright C(X) \longmapsto C(X) \rtimes \Gamma = C^*(X \rtimes \Gamma)$

$\hat{\Gamma} \overset{\text{triv}}{\circlearrowright} C(X) \longmapsto C(X) \otimes C(\Gamma) = C(X \rtimes \Gamma)$

braided-commuative quantum dynamical system $\hat{\Gamma}, \Gamma \circlearrowright N$ $\overset{?}{\longrightarrow}$ **dual pair of quantum groupoids** $N \rtimes \Gamma, N \rtimes \hat{\Gamma}$

Which structure is present and what are we looking for?

crossed product

- algebras
  $N \hookrightarrow N \rtimes \Gamma$
- dual coaction
  $N \rtimes \Gamma \to N \rtimes \Gamma \otimes \mathbb{C}\Gamma$
- cond. expectation
  $N \rtimes \Gamma \to N$

quantum groupoid

- algebras
  $N \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A$
- comultiplication
  $A \xrightarrow{\Delta} A_{\beta} *_{\alpha} A$
- Haar weights
  $A \overset{\phi}{\underset{\psi}{\rightrightarrows}} N$
- antipode
  $A \supseteq A_0 \to A$
- Pontrjagin dual

groupoid

- spaces
  $X \overset{r}{\underset{s}{\leftleftarrows}} G$
- multiplication
  $G \leftarrow G_s \times_r G$
- Haar systems
- inversion
  $G \leftarrow G$

Reverse-engineering the requirements on $\Gamma \circlearrowright N$

Assume $\Gamma \circlearrowright N$. Given $N \underset{\beta\colon y \mapsto \sum_\gamma y_\gamma \gamma}{\overset{\alpha\colon x \mapsto xe}{\rightrightarrows}} A = N \rtimes \Gamma$, observe

1. $[\alpha(x), \beta(y)] = 0$ if and only if $xy_\gamma = y_\gamma \gamma(x)$ for all γ
2. $\beta(y'y) = \beta(y)\beta(y') = \sum_{\gamma,\gamma'} y_\gamma \gamma(y'_{\gamma'})\gamma\gamma' = \sum_{\gamma,\gamma'} y'_{\gamma'} y_\gamma \gamma\gamma'$ if and only if $\tilde{\beta}\colon N \to N \otimes \mathbb{C}\Gamma, y \mapsto \sum_\gamma y_\gamma \otimes \gamma^{-1}$ is a homomorphism
3. $\Delta\colon N \rtimes \Gamma \xrightarrow{x\gamma \mapsto x\gamma\otimes\gamma} N \rtimes \Gamma \otimes \mathbb{C}\Gamma \to (N \rtimes \Gamma)_{\beta} *_{\alpha} (N \rtimes \Gamma)$ satisfies
   - $\alpha(x) \mapsto \alpha(x) \otimes 1$ always
   - $\beta(y) \mapsto 1 \otimes \beta(y)$ if and only if $\tilde{\beta}$ is a coaction.

Summary: $N \rtimes \Gamma$ becomes a bialgebroid with respect to α, β, Δ if and only if N is a braided-commutative Γ-Yetter-Drinfeld algebra

Quantum transformation groupoids in the algebraic setting

Theorem (Lu; Brzezinski-Militaru) Let H be a Hopf algebra acting on an algebra N. Then $N \rtimes H$ becomes a Hopf algebroid if and only if N is a braided-commutative H-Yetter-Drinfeld algebra.

Examples

1. *Commutative super-algebras:* $N = N_0 \oplus N_1$ and $H = \mathbb{C}\mathbb{Z}_2$, where $ab = (-1)^{\deg(a)\cdot\deg(b)} ba$ for all $a, b \in N$
2. *The quantum plane:* $N = \mathbb{C}_q[x,y]$ and $H = \mathcal{O}(\mathrm{GL}_q(2))$, where

   $\mathbb{C}_q[x,y] = \langle x, y : yx = qxy \rangle$,

   $\mathcal{O}(\mathrm{GL}_q(2)) = \left\langle a,b,c,d : \begin{smallmatrix} ba=qab, & ca=qac, & db=qbd, & dc=qcd \\ bc=cb, & ad-q^{-1}bc=da-qbc \text{ invertible} \end{smallmatrix} \right\rangle$,

   $N \to N \otimes H$ s.t. $(x,y) \mapsto (x,y) \boxtimes \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$,

   $H \otimes N \to N$ s.t. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \boxtimes x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & q^{-1}x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \boxtimes y \mapsto \left(\begin{smallmatrix} qy & 0 \\ 0 & y \end{smallmatrix}\right)$

Yetter-Drinfeld algebras in the setting of operator algebras

Let (M, Δ) be a locally compact quantum group with dual $(\hat{M}, \hat{\Delta})$

Definition (Nest-Voigt) A Yetter-Drinfeld algebra over (M, Δ) is a von-Neumann algebra N with coactions α of M and λ of $\hat{M}$ s.t.

(a)
$$\begin{array}{ccc} N & \xrightarrow{(\iota\otimes\lambda)\circ\alpha} & M \otimes \hat{M} \otimes N \\ \| & \circlearrowleft & \downarrow (\sigma\otimes\iota)\circ(\mathrm{ad}_W \otimes \iota) \\ N & \xrightarrow[(\iota\otimes\alpha)\circ\lambda]{} & \hat{M} \otimes M \otimes N \end{array}$$

where $W \in M \otimes \hat{M}$ is the multiplicative unitary and σ the flip

or, equivalently

(b) $(\iota \otimes \lambda) \circ \alpha$ is a coaction of the quantum double $D(M) = M \otimes \hat{M}$

Example left coideals $N \subseteq D(M)$ with regular coaction, e.g. $M = \mathcal{O}(\mathrm{SU}_q(2))''$ and $N = \mathcal{O}(S_q^2)'' = \mathcal{O}(\mathrm{SU}_q(2)/T)'' \subseteq M$

The setup

Setup Fix (N, α, λ) with some weight ν and GNS-rep. $N \circlearrowright H_\nu$.

We shall need the unitary implementations (Vaes)

- $X \in M \otimes \mathcal{L}(H_\nu)$ satisfying $\alpha(x) = X(1 \otimes x)X^*$
- $Y \in \hat{M} \otimes \mathcal{L}(H_\nu)$ satisfying $\lambda(x) = Y(1 \otimes x)Y^*$

for example

- if ν α-invariant, bounded, then $X(\xi \otimes \Lambda_\nu(x)) = \alpha(x)(\xi \otimes \Lambda_\nu(1))$
- if $(N, \alpha) = (M, \Delta)$ and ν is the left Haar weight, then $X = W^*$

Assumption The following equivalent conditions hold:

1. $Y_{23}X_{13} \in M \otimes \hat{M} \otimes \mathcal{L}(H_\nu)$ is a corepresentation of $D(M)$
2. $Y_{23}X_{13} = W^*_{12}X_{13}Y_{23}W_{12}$

Examples The assumption holds if

1. $(\iota \otimes \lambda) \circ \alpha$ is a dual action of $D(M)$ or
2. ν is suitably invariant

The crossed product $M \ltimes N$ as a von Neumann bimodule

Let H be the L^2-space of (M, Δ) and $(\hat{M}, \hat{\Delta})$ and consider

- the crossed product $M \ltimes_\alpha N = ((\hat{M} \otimes 1) \cup \alpha(N))'' \subseteq \mathcal{L}(H \otimes K)$
- the inclusion $\alpha \colon N \to M \ltimes_\alpha N$, $x \mapsto \mathrm{ad}_X(1 \otimes x)$
- the anti-rep. $\beta \colon N \to \mathcal{L}(H \otimes H_\nu)$, $x \mapsto \mathrm{ad}_{X(J \otimes J_\nu)Y}(1 \otimes x^*)$

Lemma $[\alpha(N), \beta(N)] = \mathrm{ad}_X([1 \otimes N, \mathrm{ad}_{(J \otimes J_\nu)}(\lambda(N))]) = 0$

Lemma The following conditions are equivalent:

1. $\beta(N) \subseteq M \ltimes_\alpha N$, **2.** $[\alpha'(N'), \lambda'(N')] = 0$, **3.** $[\alpha^{\mathrm{op}}(N'), \lambda^{\mathrm{op}}(N')] = 0$,
where α', λ' are the commutants and $\alpha^{\mathrm{op}}, \beta^{\mathrm{op}}$ the opposites, obtained from α, λ by conjugating with $J \otimes J_\nu$ or $\hat{J} \otimes J_\nu$

Definition (N, α, λ) is braided-commutative if conditions 1.-3. hold

Proof 2.⇔3. because $\mathrm{ad}_{(J \otimes J_\nu)}$ and $\mathrm{ad}_{(\hat{J} \otimes J_\nu)}$ commute
1.⇔3. $\beta(N) = \mathrm{ad}_X(\lambda^{\mathrm{op}}(N')) \subseteq \mathrm{ad}_X(\hat{M} \otimes N')$ commutes with
$\mathrm{ad}_X(\alpha^{\mathrm{op}}(N')) \vee \mathrm{ad}_X(\hat{M}' \otimes 1) = \mathrm{ad}_{X(\hat{J} \otimes J_\nu)}(M \ltimes_\alpha N) \overset{\text{(Vaes)}}{=} (M \ltimes_\alpha N)'$

The comultiplication, left-invariant weight and the dual

We obtained a von Neumann bimodule $N \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M \ltimes_\alpha N \circlearrowright L := H \otimes K$.

The unitary implementation X leads to a canonical unitary

$$Z \colon (H \otimes H_\nu) \otimes H \to (H \otimes H_\nu)_\beta \underset{\nu}{\otimes}_\alpha (H \otimes H_\nu) =: L_\beta \underset{\nu}{\otimes}_\alpha L$$

Theorem

1. $A := M \ltimes_\alpha N$ is a Hopf-von Neumann bimodule w.r.t. α, β and $\Delta_A \colon A \xrightarrow{\text{dual coaction } \hat{\alpha}} \hat{M} \otimes A \subseteq \mathcal{L}(H \otimes L) \xrightarrow{\mathrm{ad}_{Z\Sigma}} \mathcal{L}(L_\beta \underset{\nu}{\otimes}_\alpha L)$, i.e.,
   - $\Delta_A(\alpha(x)) = \alpha(x) \otimes 1$
   - $\Delta_A(\beta(x)) = 1_\beta \underset{\nu}{\otimes}_\alpha \beta(x)$
   - $\Delta_A(A) \subseteq A_\beta \underset{\nu}{*}_\alpha A = (A'_\beta \underset{\nu}{\otimes}_\alpha A')'$
   - $(\Delta_A * \iota)\Delta_A = (\iota * \Delta_A)\Delta_A$
2. $T_L = (\hat{\phi} \otimes \iota \otimes \iota) \circ \hat{\alpha} \colon A \to \alpha(N)$ is left-invariant w.r.t. Δ_A
3. The associated left fundamental isometry (Lesieur) is $L_\alpha \underset{\nu}{\otimes}_{\beta'} L \xrightarrow{Z'^*} H \otimes H_\nu \otimes H \xrightarrow{\hat{W}_{13}} H \otimes H_\nu \otimes H \xrightarrow{Z} L_\beta \underset{\nu}{\otimes}_\alpha L$
4. The associated dual Hopf-v.N. bimodule is $\mathrm{ad}_{XY^*}(\hat{M} \ltimes_\lambda N)$.

The co-involution or unitary antipode

Using the work of Vaes on crossed products, we

- consider the dual weight $A = M \ltimes_\alpha N \xrightarrow{T_L} \alpha(N) \cong N \xrightarrow{\nu} \mathbb{C}$
- identify $L^2(A)$ with $H \otimes H_\nu$ via $(y \otimes 1)\alpha(x) \mapsto \hat{\Lambda}(y) \otimes \Lambda_\nu(x)$
- obtain by a polar decomposition of the involution on $L^2(A)$ the modular operator ∇_A and conjugation $J_A = X(\hat{J} \otimes J_\nu)$
- replace $A = M \ltimes_\alpha N$ by $\hat{A} = \mathrm{ad}_{XY^*}(\hat{M} \ltimes_\lambda N)$ and obtain $\nabla_{\hat{A}}, J_{\hat{A}}$

Proposition Define $R_A \colon A \to \mathcal{L}(H \otimes H_\nu)$ by $z \mapsto J_{\hat{A}} z^* J_{\hat{A}}$. This map

1. is a co-involution on the Hopf-v.N. bimodule $A = M \ltimes_\alpha N$
2. satisfies strong invariance w.r.t. T_L
3. yields a right-invariant $T_R := R_A \circ T_L \circ R_A \colon A \to \beta(N)$

Summary: What we get and what is needed

Theorem (T.) Let (N, α, λ) be a braided-commutative Yetter-Drinfeld algebra. Then we obtain

- a Hopf-von Neumann bimodule $N \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M \ltimes_\alpha N = A \xrightarrow{\Delta_A} A_\beta \underset{\nu}{*}{}_\alpha A$ with a co-involution R_A
- left-/right-invariant weights $T_L = (\hat{\phi} \otimes \iota) \circ \hat{\alpha}$, $T_R = R_A \circ T_L \circ R_A$

Problem To obtain a measured quantum groupoid, we need a n.s.f. weight ν on N such that the modular automorphism groups of $\nu \circ \alpha^{-1} \circ T_L$ and $\nu \circ \beta^{-1} \circ T_R$ on $M \ltimes_\alpha N$ commute.