Tannaka-Krein duality for partial compact quantum groups and the dynamical $SU_q(2)$

(joint work with Kenny De Commer)

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18th of November 2014

Classical Tannaka-Krein duality

For a compact group G, its category $\mathrm{Rep}_{\mathrm{u,fd}}(G)$

(i) is semi-simple (every object is a direct sum of irreducibles)

(ii) carries a tensor product which is symmetric ($\pi \otimes \pi' \cong \pi' \otimes \pi$)

(iii) has a faithful tensor functor, briefly fiber functor, F into Hilb

(iv) has a certain duality (contragredient representations)

Theorem (Tannaka-Krein) Every category $\mathcal{C}$ satisfying (i)–(iv) is equivalent to $\mathrm{Rep}_{\mathrm{u,fd}}(G)$ for some compact group G

Idea The group G consists of all families $\eta \in \prod_X \mathcal{B}(FX)$ such that

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circlearrowright"] & FY \arrow[d, "\eta_Y"] \\ FX \arrow[r, "Ff"] & FY \end{tikzcd} , \quad \text{each } \eta_X \text{ is unitary}, \quad \eta_{X \otimes Y} = \eta_X \otimes \eta_Y$$

Every $X \in \mathcal{C}$ yields a unitary representation $G \to \mathcal{B}(FX)$, $\eta \mapsto \eta_X$

Duality in a tensor category, and $\mathsf{Rep}_{\mathsf{u,fd}}(\mathsf{SU}(2))$

Definition Objects X and $\overline{X}$ in a strict tensor category are dual if there are morphisms $R\colon 1 \to X \otimes \overline{X}$ and $R^\dagger\colon \overline{X} \otimes X \to 1$ such that

$$X \xrightarrow{R \otimes \mathrm{id}} X \otimes \overline{X} \otimes X \xrightarrow{\mathrm{id} \otimes R^\dagger} X \text{ and } \overline{X} \xrightarrow{\mathrm{id} \otimes R} \overline{X} \otimes X \otimes \overline{X} \xrightarrow{R^\dagger \otimes \mathrm{id}} \overline{X}$$

are the identity.

Example For $G = \mathrm{SU}(2)$, the category $\mathrm{Rep}_{\mathrm{u,fd}}(\mathrm{SU}(2))$ has

- the fundamental irrducible representation u on $\mathbb{C}^2$ and irreducibles $u_k = \mathrm{Sym}^k u$ labelled by integers $k \in \mathbb{N}$
- tensor product $u_k \otimes u_l \cong u_{|k-l|} \oplus \cdots \oplus u_{k+l}$
- duality morphisms $R\colon 1 \to u \otimes u$ and $R^*\colon u \otimes u \to 1$, i.e. $\overline{u} = u$, which generate all morphisms in $\mathrm{Rep}_{\mathrm{u,fd}}(\mathrm{SU}(2))$
- a purely combinatorial description of each $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$

The proof of Tannaka-Krein makes crucial use of algebras

Given $\mathcal{C} \xrightarrow{F} \mathrm{Hilb}$, define $G \subseteq \prod_X \mathcal{B}(FX)$ and $\pi_X\colon \eta \mapsto \eta_X$ as above.

Each object $X \in \mathcal{C}$ and functional $\omega \in \mathcal{B}(FX)_*$ yield a function

$$f^X_\omega = \omega \circ \pi_X\colon G \to \mathbb{C}, \quad \eta \mapsto \omega(\eta_X).$$

Then the set $\mathcal{T}(G) \subseteq C(G)$ of such functions

- is a subspace ($f^X_\upsilon + f^Y_\omega = f^{X\oplus Y}_{\upsilon\oplus\omega}$) an algebra ($f^X_\upsilon \cdot f^Y_\omega = f^{X\otimes Y}_{\upsilon\otimes\omega}$), and a $*$-algebra, as can be seen using dual objects
- separates the points of G and therefore is dense in $C(G)$
- consists of all matrix elements of f.d. representations (Peter-Weyl)

Hence, $\mathcal{C} \mapsto \mathrm{Rep}_{\mathrm{u,fd}}(G)$, $X \mapsto \pi_X$, is essentially surjective on objects.

The compact quantum groups of Woronowicz

By Gelfand-Naimark duality, every compact group G is determined by its function algebra and the transpose of the multiplication

$$\Delta\colon C(G) \to C(G \times G) \cong C(G) \otimes C(G)$$

Definition (Woronowicz) A compact quantum group is a unital C^*-algebra A with a $*$-homomorphism $\Delta\colon A \to A \otimes A$ satisfying

(i) coassociativity: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$, and

(ii) cancellation: $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$

Example A is commutative iff $A \cong C(G)$ for some compact space G

- Δ corresponds to an associative multiplication $G \times G \to G$
- cancellation holds iff two maps $G \times G \to G \times G$ are injective:
  $$(x, y) \mapsto (xy, y) \text{ and } (x, y) \mapsto (x, xy)$$
- a compact semigroup with cancellation is a group

Examples of compact quantum groups

The following are compact matrix quantum groups of the form

$$A = \langle u_{ij} \mid u = (u_{ij})_{i,j} \text{ is unitary} \rangle, \qquad \Delta(u_{ij}) = \textstyle\sum_k u_{ik} \otimes u_{kj}$$

- q-deformation of polynomial functions on SU(2)
  $$\mathcal{O}(\mathrm{SU}(2)) = \left\langle a, c \,\middle|\, u = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \text{ is unitary} \right\rangle$$
- The quantum permutation group
  $$A = \left\langle p_{ij}, 1 \leq i, j \leq n \,\middle|\, \begin{array}{l} \text{each row/column of } (p_{ij})_{i,j} \text{ consists of} \\ \qquad\qquad \text{pairw. orthog. projections with sum 1} \end{array} \right\rangle$$
- The free orthogonal quantum group for parameter $F \in \mathrm{GL}_n(\mathbb{C})$
  $$A_o(F) = \langle u_{ij}, 1 \leq i, j \leq n \mid F\bar{u} = uF \rangle$$

Tannaka-Krein-Woronowicz duality

Every compact quantum group $\mathbb{G} = (A, \Delta)$ has a category $\mathrm{Rep}_{\mathrm{fd}}(\mathbb{G})$ of representations on f.d. Hilbert spaces, where

- a representation of $\mathbb{G}$ on a f.d. Hilbert space H is an invertible $X \in A \otimes \mathcal{B}(H)$ satisfying $(\Delta \otimes \mathrm{id})(X) = X_{13}X_{23}$
- a morphism of representations X, Y on Hilbert spaces H, K is a $T \in \mathcal{B}(H, K)$ satisfying $(1 \otimes T)X = Y(1 \otimes T)$
- the tensor product of X and Y is $X_{12}Y_{13} \in A \otimes \mathcal{B}(H \otimes K)$
- the dual of X is $j(X) \in A \otimes \mathcal{B}(\overline{H})$, where $j(a \otimes b) = a^* \otimes \overline{b}$

Theorem (Woronowicz) Every semi-simple tensor C^*-category $\mathcal{C}$ with duality and a fiber functor to Hilb is equivalent to $\mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G})$ for some compact quantum group $\mathbb{G}$.

Idea $A_0 := \bigoplus_{X \in \mathrm{Irr}(\mathcal{C})} \mathcal{B}(FX)_*$ is a Hopf $*$-algebra and $A = C^*(A_0)$

Passing from (quantum) groups to (quantum) groupoids

- a set G
- a map $G \times G \xrightarrow{m} G$
- conditions . . .

- sets X and G
- maps $G_s\times_r G \xrightarrow{m} G \overset{r}{\underset{s}{\rightrightarrows}} X$
- conditions . . .

$$\begin{tikzcd}
\text{group} \arrow[r, dashed] \arrow[d, dashed] & \text{groupoid} \arrow[d, dashed] \\
\text{quantum group} \arrow[r, dashed] & \text{quantum groupoid}
\end{tikzcd}$$

- an algebra A
- a map $A \xrightarrow{\Delta} A \otimes A$
- conditions . . .

- algebras B and A
- maps $B \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \xrightarrow{\Delta} A_\beta \times_\alpha A$
- conditions . . .

Passing to quantum groupoids≡replacing the target category

We now consider tensor functors into ${}_I\mathrm{Hilb}_I$, where I is a set and

- objects are families of Hilbert spaces $\mathcal{H} = ({}_kH_l)_{k,l\in I}$
- morphisms are families of linear operators
- the tensor product of $\mathcal{H}$ and $\mathcal{K}$ is $\mathcal{H} \underset{I}{\otimes} \mathcal{K} = \left(\bigoplus_{l} {}_kH_l \otimes {}_lK_m\right)_{k,m}$

Examples

- The canonical fiber functor of a tensor C^*-category (Hayashi)
  Given such a category $\mathcal{C}$, write $\mathrm{Irr}(\mathcal{C}) = (u_k)_{k\in I}$ and define
  $$F\colon \mathcal{C} \to {}_I\mathrm{Hilb}_I,\ \ X \mapsto \big(\mathrm{Hom}(u_k, X \otimes u_l)\big)_{k,l}$$
  Get $\mathrm{Hom}(k, X \otimes l) \otimes \mathrm{Hom}(l, Y \otimes m) \to \mathrm{Hom}(k, X \otimes Y \otimes m)$ and
  $$FX \underset{I}{\otimes} FY \to F(X \otimes Y).$$
- Monoidal equivalence of CQGs (Bichon–De Rijdt–Vaes)
  Given $\mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G}_1) \sim \mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G}_2)$, write $I = \{1,2\}$ and obtain
  $$\mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G}_1) \to \mathrm{Hilb} \times \mathrm{Hilb} \hookrightarrow {}_I\mathrm{Hilb}_I$$

“Partial” Tannaka-Krein-Woronowizc duality

Theorem (De Commer–T.) There exists a correspondence, bijective up to equivalence, between connected partial compact quantum groups $\mathbb{G}$ and semi-simple tensor C^*-categories with duality and fiber functor to ${}_I\mathrm{Hilb}_I$, given by $\mathbb{G} \mapsto \mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G})$.

Ideas A connected partial CQG $\mathbb{G}$ is given by spaces ${}^k_mA^l_n$ with

- multiplications ${}^k_mA^l_n \times {}^l_nA^p_q \to {}^k_mA^p_q$ and involutions ${}^k_mA^l_n \to {}^l_nA^k_m$ that turn $P(\mathbb{G}) = \bigoplus_{k,l,m,n} {}^k_mA^l_n$ into a $*$-algebra with units in ${}^k_mA^k_m$
- comultiplications $\Delta_{pq}\colon {}^k_mA^l_n \to {}^k_pA^l_q \otimes {}^p_mA^q_n$, counits and antipodes that turn $P(\mathbb{G})$ into a weak multiplier Hopf $*$-algebra
- a Haar weight $\phi\colon P(\mathbb{G}) \to \mathbb{C}$ that is positive, faithful,, invariant

From $F\colon \mathcal{C} \to {}_I\mathrm{Hilb}_I$, get $\mathbb{G}$ via ${}^k_mA^l_n = \bigoplus_{X\in\mathrm{Irr}(\mathcal{C})} \mathcal{B}({}_m(FX)_n, {}_k(FX)_l)_*$.

Partial quantum groups on the level of operator algebras

Given a partial compact quantum group $\mathbb{G}$, we construct the following completions of the polynomial algebra $P(\mathbb{G})$:

- a universal C^*-algebra $C^u(\mathbb{G}) = C^*(P(\mathbb{G}))$ with comultiplication
- a reduced C^*-algebra $C^r(\mathbb{G})$ and von Neumann algebra $L^\infty(\mathbb{G})$, generated by the regular representation $\pi_r\colon P(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$, with lifts of the comultiplication and of the Haar weight
  - define $L^2(\mathbb{G})$ as a completion of $P(\mathbb{G})$ using the Haar weight, need to prove boundedness of $\pi_r(a)\colon b \mapsto ab$ on $L^2(\mathbb{G})$
  - use a partial isometry $V\colon a \otimes b \mapsto \Delta(a)(1 \otimes b)$ on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$, then each $\pi_r(a)$ arises as $(\omega \otimes \mathrm{id})(V)$ for some $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ and the comultiplication lifts by the formula $x \mapsto V(1 \otimes x)V^*$
  - for the Haar weight, use that $P(\mathbb{G}) \subseteq L^2(\mathbb{G})$ is a Hilbert algebra

Theorem $L^\infty(\mathbb{G})$ is a measured quantum groupoid (Enock, Lesieur).

Application to the dynamical $\mathrm{SU}_q(2)$

Note that $S^2 \cong \mathrm{SU}(2)/\mathbb{T}$ is a homogeneous space for $\mathrm{SU}(2)$.
Podleś discovered quantum homogeneous spaces $S^2_{q,x}$ for $\mathrm{SU}_q(2)$.

Theorem (De Commer–Yamashita) For every compact quantum group $\mathbb{G}$, there exists a bijective correspondence between

(i) quantum homogeneous spaces for $\mathbb{G}$
(ii) connected, semi-simple C^*-module categories $\mathcal{D}$ over $\mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G})$
(iii) connected fiber functors from $\mathrm{Rep}_{\mathrm{u,fd}}(\mathbb{G})$ to ${}_I\mathrm{Hilb}_I$ for some I

Thus $S^2_{q,x}$ corresponds to some $F\colon \mathrm{Rep}_{\mathrm{u,fd}}(\mathrm{SU}_q(2)) \to {}_I\mathrm{Hilb}_I$ and to a partial compact quantum group $\mathbb{G}_{q,x}$, which turns out to be a variant of the dynamical $\mathrm{SU}_q(2)$ (Koelink–Rosengren)

Theorem For $q \neq 1$, this partial compact quantum group $\mathbb{G}_{q,x}$ is not coamenable in the sense that $C^u(\mathbb{G}_{q,x}) \to C^r(\mathbb{G}_{q,x})$ is not injective