FROM TANNAKA-KREIN DUALITY TO QUANTUM GROUPOIDS

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CLASSICAL TANNAKA-KREIN DUALITY

Let G be a compact group. The category of finite-dimensional unitary representations of G has the following properties:

1. It is *semi-simple*.
   (every representation is a direct sum of irreducible ones)
2. It has a *tensor product*.
   (given $G \circlearrowright H, K$, get $G \circlearrowright H \otimes K$, where $g(\xi \otimes \eta) = g\xi \otimes g\eta$)
3. This product is *symmetric*.
   (for H, K as above, the flip $H \otimes K \to K \otimes H$ intertwines G)
4. For each representation, there exists a *dual* one.
   (given $G \circlearrowright H$, get $G \circlearrowright \overline{H}$, where $g\overline{\xi} := \overline{g\xi}$)
5. It has a *tensor functor to* the category of *Hilbert spaces*.

Theorem (Tannaka-Krein) There exists a duality

$$\text{(compact groups)} \quad \stackrel{\sim}{\longleftrightarrow} \quad \text{(categories satisfying 1.–5.)}$$

RECONSTRUCTION IN FINITE CASE

Let $\mathcal{C}$ and $F\colon \mathcal{C} \to \textbf{\textit{Hilb}}$ satisfy 1.–5. and assume $|\operatorname{Irr}(\mathcal{C})| < \infty$.

1.$\rightsquigarrow$ $\mathcal{C} \sim {}_A\textbf{\textit{Mod}}$, where $A = \bigoplus_{[X]\in\operatorname{Irr}(\mathcal{C})} \operatorname{End}(FX)$

2.,5.$\rightsquigarrow$ use ${}_AA \otimes {}_AA \in {}_A\textbf{\textit{Mod}}$ to define $\Delta\colon A \to A \otimes A$, $a \mapsto a \cdot (1_A \otimes 1_A)$

2.,5.$\rightsquigarrow$ use tensor unit $\mathbf{1} \in {}_A\textbf{\textit{Mod}}$ and $F\mathbf{1} = \mathbb{C}$ to define $\varepsilon\colon A \to \mathbb{C}$, $a \mapsto a \cdot 1_{\mathbb{C}}$

3.$\rightsquigarrow$ Δ is *cocommutative*

4.$\rightsquigarrow$ A has the structure of a *Hopf algebra*

Finally, ${}_A\textbf{\textit{Mod}} \cong \textbf{\textit{Rep}}(G)$, where $G := \{x \in A \text{ invertible} : \Delta(x) = x \otimes x\}$.

COMPACT QUANTUM GROUPS

Definition (Woronowicz)

- A *compact quantum group* is a unital C^*-algebra A [1] with a $*$-homomomorphism $\Delta\colon A \to A \otimes A$ such that
  1. *(coassociativity)*
  $$\begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & A \otimes A \\ {\scriptstyle\Delta}\downarrow & \circlearrowright & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ A \otimes A & \xrightarrow[\Delta\otimes\mathrm{id}]{} & A \otimes A \otimes A \end{array}$$
  2. $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ are dense in $A \otimes A$ *(cancellation)*
- A *f.d. unitary representation* of (A, Δ) is a unitary $u \in M_n(A)$ satisfying $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

[1] A C^*-algebra is a unital Banach $*$-algebra satisfying $\|a^*a\| = \|a\|^2$ for all a.

CLASSICAL COMPACT GROUPS

(Compact groups)	$\longleftrightarrow$	*(Compact quantum groups* (A,Δ) *with* A *commutative)*
G compact group	$\to$	$A := C(G)$ with $\Delta(f)(x,y) = f(xy)$, where $C(G)\otimes C(G) \cong C(G\times G)$
$G := \mathrm{Hom}(A,\mathbb{C})$ with $\chi\cdot\chi' := (\chi\otimes\chi')\circ\Delta$	$\leftarrow$	(A,Δ)

A representation is a unitary $u \in M_n(C(G)) \cong C(G; M_n(\mathbb{C}))$ s.t.

$$u_{ij}(xy) = (\Delta(u_{ij}))(x,y) = \sum_k (u_{ik}\otimes u_{kj})(x,y) = \sum_k u_{ik}(x)u_{kj}(y),$$

that is, $u(xy) = u(x)u(y)$ for all $x, y \in G$.

EXAMPLE $\mathrm{SU}_Q(2)$ AND WORONOWICZ-T.K. DUALITY

The key example is $\mathrm{SU}_q(2)$, where $q \in (0,1]$:

- $C^*\left(\alpha,\gamma:\ \text{the matrix } u := \begin{pmatrix}\alpha & -q\gamma^* \\ \gamma & \alpha^*\end{pmatrix} \text{ is unitary}\right)$
- $\Delta(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}$, i.e., u is a representation

Representations $\mathrm{Irr}(\textit{Rep}(\mathrm{SU}_q(2))) = \{u_0, u_1, u_2, \ldots\}$ with

1. $\dim u_k = k+1$
2. $u_k \otimes u_l \cong u_{|k-l|} \oplus u_{|k-l|+2} \oplus \cdots \oplus u_{k+l}$

Theorem (Woronowicz) There exists a duality

(compact quantum groups) $\longleftrightarrow$ (rigid semi-simple tensor categories with a fibre functor to ***Hilb***)

LINK TO KNOT INVARIANTS

Definition A *braiding* on a monoidal category $\mathcal{C}$ is a family of natural isomorphisms $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ s.t. for all X, Y, Z,

1. $c_{X,Y\otimes Z}$ is $X \otimes Y \otimes Z \xrightarrow{c_{X,Y}\otimes \mathrm{id}} Y \otimes X \otimes Z \xrightarrow{\mathrm{id}\otimes c_{X,Z}} Y \otimes Z \otimes X$,
2. $c_{X\otimes Y,Z}$ is $X \otimes Y \otimes Z \xrightarrow{\mathrm{id}\otimes c_{Y,Z}} X \otimes Z \otimes Y \xrightarrow{c_{X,Z}\otimes \mathrm{id}} Z \otimes X \otimes Y$.

It is called *symmetric* if $c_{Y,X} = c_{X,Y}^{-1}$ for all $X, Y \in \mathcal{C}$.

Consequence Let $X \in \mathcal{C}$ and $c := c_{X,X}$. Then

$$(c \otimes \mathrm{id})\underbrace{(\mathrm{id}\otimes c)(c \otimes \mathrm{id})}_{=c_{X,X\otimes X}} = \underbrace{(\mathrm{id}\otimes c)(c \otimes \mathrm{id})}_{=c_{X,X\otimes X}}(\mathrm{id}\otimes c).$$

$\Rightarrow$ get $B_n \to \mathrm{Aut}(X^{\otimes n})$, where B_n =braid group on n strands

Proposition ***Rep***$(\mathrm{SU}_q(2))$ has a braiding c.

QUANTUM GROUPOIDS

- Consider an action of a finite group Γ on a space X.
- The Hilbert bundles on X form a tensor category ***Hilb***^{X}.
- The Γ-equivariant f.d. Hilbert bundles on X form a symmetric monoidal rigid C^*-tensor category ***Rep***$(\Gamma \circlearrowright X)$.
- Forgetting Γ, we get a functor ***Rep***$(\Gamma \circlearrowright X) \to$ ***Hilb***^{X}.
- TK-duality: can reconstruct Γ and its action from this functor.

Theorem (De Commer, T.) There exists a duality between

1. semi-simple rigid partial C^*-tensor categories $\mathcal{C}$ with local tensor units e_α ($\alpha \in I$) and a faithful functor $F \colon \mathcal{C} \to {}_I\mathbf{Hilb}_I$ satisfying ${}_iF(X \otimes Y)_j \cong \bigoplus_k {}_iF(X)_k \otimes {}_kF(X)_j$
2. *partial compact quantum groups* over I