Pseudo-multiplicative unitaries and pseudo-Kac systems on C^*-modules

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Plan

Pseudo-multiplicative unitaries – Definition

A	C^*-algebra,
E_A	C^*-module,
π_s, π_r	commut. reps. of A on E

Definition

$E \,{}_{\pi_s}\!\otimes E \xrightarrow{V} E \otimes_{\pi_r} E$ is a *pseudo-multiplicative* unitary iff

- intertwining relations for V and π_s, π_r
- $V_{12}V_{13}V_{23} = V_{23}V_{12}$ (pentagon equation)

"C^*-algebraic analogue of
pseudo-multiplicative unitaries on Hilbert spaces"

Pseudo-multiplicative unitaries – Examples

Examples

1. $A = \mathbb{C}$, $\pi_s, \pi_r =$ scalar mult. $\Rightarrow$ multiplicative unitary
   [*Baaj, Skandalis*]
2. $A = C_0(X)$, $\pi_s, \pi_r =$ right mult. $\Rightarrow$ cnt. field of mult. unit.
   [*Blanchard*]
3. G l.c. Hausdorff groupoid, λ left Haar system
   $A := C_0(G^0)$, $E := L^2(G, \lambda)$, π_s, π_r from $s, r \colon G \to G^0$
   $(Vf)(x, y) := f(x, x^{-1}y)$ def. $\overline{C_c(G\,{}_s\!\times_r G)} \to \overline{C_c(G\,{}_r\!\times_r G)}$
   [*O'uchi*]
4. C^*-analogue of groupoïde quantique espace quantique
   [*Lesieur*] (A non-commutative)

Pseudo-multiplicative unitaries – Motivation

Dualities for locally compact (quantum) groupoids

1. What is the "generalised Pontrjagin dual" of a l.c. groupoid?
2. What is a locally compact quantum groupoid?

l.c. qntm group Hopf C^*-algebra + Haar weight	$\longrightarrow$	multiplicative unitary	$\longrightarrow$	dual pair of Hopf C^*-algs
measurable qntm groupoid	$\longrightarrow$	pseudo-m. unitary on Hilbert spaces	$\longrightarrow$	dual pair of Hopf W^*-bimodules

Theorem (T '05)

pseudo-m. unitary on C^*-modules (decomposable, regular) $\rightsquigarrow$ *dual pair of Hopf C^*-families*

3. Baaj-Skandalis-Takesaki-Takai duality for l.c. groupoids?

From pseudo-m. unitaries to Hopf C^*-families

The Baaj-Skandalis construction

multiplicative unitary

$$E \otimes E \xrightarrow{V} E \otimes E$$

$\swarrow$ slice maps $\searrow$

left leg $\hat{S}$ | right leg S

$(\mathrm{id} \otimes \omega)(V),\ \omega \in B(E)_*$ | $(\omega \otimes \mathrm{id})(V),\ \omega \in B(E)_*$

$\|$ $\quad \langle\eta| - \xi\rangle \qquad \xi, \eta \in E$

$$E \xrightarrow{-\otimes|\xi\rangle} E \otimes E \xrightarrow{V} E \otimes E \xrightarrow{-\otimes\langle\eta|} E,$$

$$\zeta \mapsto \zeta \otimes \xi \qquad \zeta' \otimes \zeta'' \mapsto \zeta'\langle\eta|\zeta''\rangle$$

From pseudo-m. unitaries to Hopf C^*-families

The Baaj-Skandalis construction

pseudo-multiplicative unitary on C^*-modules

$$E \,{}_{\pi_s}\!\otimes\, E \xrightarrow{V} E \otimes_{\pi_r} E$$

$\swarrow$ slice maps $\searrow$

left leg $\hat{S}$

right leg S

?$(\mathrm{id} \otimes \omega)(V),\ \omega \in B(E)_*$?

?$(\omega \otimes \mathrm{id})(V),\ \omega \in B(E)_*$?

$\omega = \langle \eta | - \xi \rangle \qquad \xi, \eta \in E$

$$E \xrightarrow{-\otimes|\xi\rangle} E \,{}_{\pi_s}\!\otimes\, E \xrightarrow{V} E \otimes_{\pi_r} E \xrightarrow{-\otimes\langle\eta|} E,$$

$\zeta \mapsto \zeta \,{}_{\pi_s}\!\otimes\, \xi$

not left-linear

$\zeta' \otimes_{\pi_r} \zeta'' \mapsto \zeta' \langle \eta | \zeta'' \rangle$

not right-linear

not adjointable

Twisted C^*-bimodule theory – Bimodules

E C^*-module over A with representation $A \to L_A(E)$

Definition

- $\xi \in E$ is *α-homogeneous* for a partial autom. α of A
  $:\Leftrightarrow \xi \in E\,\mathrm{Dom}(\alpha),\ \xi a = \alpha(a)\xi$ for all $a \in \mathrm{Dom}(\alpha)$
- E *decomposable* $:\Leftrightarrow$ span$\{\alpha$-homogeneous $\xi \in E\}$ dense

Examples

1. $L^2(G, \lambda)$ with π_r – each element is id_A-homogeneous
2. $L^2(G, \lambda)$ with π_s – decomposable if G r-discrete
   - $U \subset G$ a G-set $\Rightarrow$ each $\xi \in C_c(U)$ is homogeneous
3. $E := A$ is decomposable $\Leftrightarrow Z(A)A = A$

Twisted C^*-bimodule theory – Operators

Definition

$T : E \to E$ is (β, α)-*homogeneous*, α, β partial automs., iff

- twisted adjoint
  $$\exists\, T^* : E \to E : \qquad \alpha(\langle T^*\eta|\xi\rangle) = \langle \eta|T\xi\rangle, \qquad \langle T^*\eta|\xi\rangle \in \mathrm{Dom}(\alpha),$$
- twisted covariance
  $$T(b\xi) = \beta(b)T\xi,\ b \in \mathrm{Dom}(\beta), \qquad \mathrm{Im}\, T \subset \mathrm{Im}(\beta)E.$$

Facts

- $L^\beta_\alpha(E) := \{(\beta, \alpha)\text{-}\textit{homogeneous operators on } E\} \subset L(E)$
- $L^\beta_\alpha(E) \cdot L^{\beta'}_{\alpha'}(E) \subset L^{\beta\beta'}_{\alpha\alpha'}(E),\ \left(L^\beta_\alpha(E)\right)^* = L^{\beta^*}_{\alpha^*}(E)$
- $L^\beta_\alpha(E) \subset L^{\beta'}_{\alpha'}(E)$ *if* $\beta < \beta',\ \alpha < \alpha'$ – *need sheaf-like axiom*
- C^*-*equality* $\|T^*T\| = \|T\|^2$

From pseudo-m. unitaries to Hopf C^*-families

pseudo-multiplicative unitary on C^*-modules

$$E \otimes E \xrightarrow{V} E \otimes E$$

(E, π_r) decomposable ↙ ↘ (E, π_s) decomposable

left leg $\left(\hat{S}(V)^\beta_\alpha\right)_{\beta,\alpha}$ right leg $\left(S(V)^\beta_\alpha\right)_{\beta,\alpha}$

||

$$E \xrightarrow{-\otimes|\xi\rangle} E \otimes E \xrightarrow{V} E \otimes E \xrightarrow{-\otimes\langle\eta|} E, \qquad \xi, \eta \in E$$

$\zeta \mapsto \zeta_{\pi_s} \otimes \xi$ — homogeneous (β, id)

$\zeta' \otimes_{\pi_r} \zeta'' \mapsto \zeta'\langle\eta|\zeta''\rangle$ — homogeneous (id, α)

homogen. $\beta,\ \alpha$

Twisted C^*-bimodule theory – (Hopf) C^*-families

Definitions

- closed subspaces $\left(C_\alpha^\beta \subset L_\alpha^\beta(E)\right)_{\beta,\alpha}$ form a *C^*-family* iff
  1. $C_\alpha^\beta \cdot C_{\alpha'}^{\beta'} \subset C_{\alpha\alpha'}^{\beta\beta'}$
  2. $(C_\alpha^\beta)^* \subset C_{\alpha^*}^{\beta^*}$
  3. sheaf-like axiom
- *multiplier C^*-family $M(C)$ ...*
- *internal tensor product* of C^*-families D and C:
  $(D \otimes_A C)_\alpha^\gamma = \overline{\text{span}}(D_\beta^\gamma \otimes_A C_\alpha^{\beta'})$ where β, β' compatible
- *morphisms* of C^*-families (non-degenerate) ...
- *Hopf C^*-family* $(S,\ \Delta\colon S \to M(S \otimes_A S))$ ...

From pseudo-m. unitaries to Hopf C^*-families

Main result

Theorem (T '05)

V decomposable regular pseudo-m. unitary

$\Rightarrow (\hat{S},\hat{\Delta}),\ (S,\Delta)$ *Hopf C^*-families, where*

$$\hat{S}_\alpha^\beta := \overline{\mathit{span}}\left\{E \xrightarrow[\xi\ \beta\text{-homog.}]{-\otimes|\xi\rangle} E\ {}_{\pi_s}\!\otimes E \xrightarrow{V} E \otimes_{\pi_r} E \xrightarrow[\eta\ \alpha\text{-homog.}]{-\otimes\langle\eta|} E\right\}$$

$$\hat{\Delta}(\hat{s}) := V^*(1 \otimes_A \hat{s})V$$

Example

V pseudo-m. unitary associated to a l.c. Hausdorff groupoid G

- $(\hat{S},\hat{\Delta})$ – function algebra $C_0(G)$, $\quad (\hat{\Delta}f)(x,y) = f(xy)$
- (S,Δ) – left reg. representation of G, $\quad \Delta(\lambda_x) = \lambda_x \otimes \lambda_x$

Pseudo-Kac systems – Motivation

Baaj-Skandalis-Takesaki-Takai duality

Kac system = multiplicative unitary ($\rightsquigarrow$ Hopf C^*-algebras $\hat{S}, S$) + unitary antipode

– framework for generalised Takesaki-Takai duality –

coaction	$\to$	reduced dual coaction	$\to$	Baaj-Skandalis duality theorem
$C \circlearrowleft S$		$C \rtimes_r \hat{S} \circlearrowleft \hat{S}$		$C \rtimes_r \hat{S} \rtimes S \sim_M C$

Coactions of Hopf C^*-families

(S, Δ) Hopf C^*-family on E_A

C *C^*-algebra over A* (i.e. with $A \to M(C)$ non-deg.)

$\rightsquigarrow C \,{}_A\!\otimes S$ a C^*-algebra over A

Definition

- $\delta\colon C \to M(C \,{}_A\!\otimes S)$ a *coaction* of (S, Δ) :$\Leftrightarrow$
  - δ is A-equivariant
  - $\delta(C)(1 \,{}_A\!\otimes S) \subset C \,{}_A\!\otimes S$ and $(\mathrm{id} \,{}_A\!\otimes \Delta)\delta = (\delta \,{}_A\!\otimes \mathrm{id})\delta$
- (C, δ) a (S, Δ)-*algebra* :$\Leftrightarrow$ $\delta(C)(1 \,{}_A\!\otimes S) = C \,{}_A\!\otimes S$ (roughly)

Theorem (T '05)

V pseudo-m. unitary associated to a l.c. Hausdorff groupoid G

- *$(\hat{S}, \hat{\Delta})$-algebras $\cong$ actions of G on $C_0(G^0)$-algebras*
- *(S, Δ)-algebras $\cong$ usc. Fell bundles on G (G r-discrete)*

Pseudo-Kac systems – Attempted definition

The pseudo-Kac system of a groupoid

Example

G groupoid with left Haar system λ

- pseudo-multiplicative unitary: ✓
- unitary antipode: inversion $\rightsquigarrow \underbrace{L^2(G,\lambda)}_{E} \rightleftarrows \underbrace{L^2(G,\lambda^{-1})}_{F}$
  - no polar decomposition, no modular function
  - but $C_c(G_s \times_r G) \xrightarrow{V_0} C_c(G_r \times_r G) \rightsquigarrow$ additional unitaries $F \otimes_{\pi_r} F \to E \otimes_{\pi_r} E$, $F \,{}_{\pi_s}\!\otimes E \to F \,{}_{\pi_r}\!\otimes E$

Pseudo-Kac systems – Definition and examples

Definition

A *pseudo-Kac system* consists of

- C^*-modules and unitaries $E_s \xrightarrow{U} E_r$ and $E_{\overline{s}} \xrightarrow{\overline{U}} E_{\overline{r}}$
- representations $\pi_r^s, \ldots, \pi_{\overline{s}}^{\overline{r}}$
- unitaries $V^{ss}, \ldots, V^{\overline{r}\overline{r}}$

s.t. 1) pentagon equation for family V 2) ...

Examples

1. l.c. Hausdorff groupoid with left Haar system
2. groupoïde quantique espace quantique

Pseudo-Kac systems – Duality theorem

Generalised Baaj-Skandalis-Takesaki-Takai duality

decomposable pseudo-Kac system	$\rightarrow$	pair of Hopf C^*-families $\hat{S}, S$ with *initial* coactions $(\hat{S}_0, \hat{\delta}_0), (S_0, \delta_0)$
coaction of (S, Δ) (C, δ)	$\rightarrow$	*reduced dual coaction of* $(\hat{S}, \hat{\Delta})$ $\left(C \rtimes_r \hat{S}_0 = \delta(C)(1 {}_A\otimes \hat{S}_0),\ 1 {}_A\otimes \hat{\delta}_0\right)$

Theorem (T' 05)

C regular (S, Δ)-*algebra* $\Rightarrow C \rtimes_r \hat{S}_0 \rtimes_r S_0 \cong K_C(C {}_A\otimes E)$

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